

SET THEORY CHEAT SHEET

1. Basic Identities

1. Idempotent Laws

- $A \cup A = A$
- $A \cap A = A$

2. Identity Laws

- $A \cup \emptyset = A$
- $A \cap U = A$

(U = Universal set, $\emptyset$ = Empty set)

3. Domination Laws

- $A \cup U = U$
- $A \cap \emptyset = \emptyset$

4. Complement Laws

- $A \cup A^c = U$
- $A \cap A^c = \emptyset$

5. Double Complement Law

- $(A^c)^c = A$

2. Commutative Laws

1. Union Commutative

- $A \cup B = B \cup A$

2. Intersection Commutative

- $A \cap B = B \cap A$

3. Associative Laws

1. Union Associative

- $(A \cup B) \cup C = A \cup (B \cup C)$

2. Intersection Associative

- $(A \cap B) \cap C = A \cap (B \cap C)$

9. Principle of Inclusion–Exclusion (PIE)

- For two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- For three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- General case for n sets $A_1, A_2, \dots, A_n$:

$$\left| \bigcup_{i=1}^n A_i \right| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n|$$

4. Distributive Laws

1. Union over Intersection

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2. Intersection over Union

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5. Absorption Laws

1. Union Absorption

- $A \cup (A \cap B) = A$

2. Intersection Absorption

- $A \cap (A \cup B) = A$

6. De Morgan's Laws

1. Complement of Union

- $(A \cup B)^c = A^c \cap B^c$

2. Complement of Intersection

- $(A \cap B)^c = A^c \cup B^c$

7. Set Difference & Symmetric Difference

1. Set Difference

- $A - B = A \cap B^c$

2. Symmetric Difference

- $A \Delta B = (A - B) \cup (B - A)$
- Equivalent: $A \Delta B = (A \cup B) - (A \cap B)$

3. Symmetric Difference Properties

- $A \Delta A = \emptyset$
- $A \Delta \emptyset = A$
- $A \Delta B = B \Delta A$
- $(A \Delta B) \Delta C = A \Delta (B \Delta C)$ (As: $\downarrow$ ative)

8. Other Useful Laws

1. Subset Laws

- $A \subseteq B \iff A \cup B = B$
- $A \subseteq B \iff A \cap B = A$

2. Complement Laws with Universal/Empty Set

- $U^c = \emptyset$
- $\emptyset^c = U$

3. Law of Nullification

- $A \cap (A^c \cup B) = A \cap B$
- $A \cup (A^c \cap B) = A \cup B$

10. Disjointness & Partition Laws

1. Disjoint Sets

- If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.

2. Partition of Universal Set

- If $\{A_1, A_2, \dots, A_n\}$ is a partition of U , then
 - $A_i \cap A_j = \emptyset$ for $i \neq j$, and
 - $A_1 \cup A_2 \cup \dots \cup A_n = U$.

11. Cartesian Product Laws

1. Distributive with Union/Intersection

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$

2. Empty Set Rule

- $A \times \emptyset = \emptyset$
- $\emptyset \times A = \emptyset$

12. Power Set Properties

1. Definition: $\mathcal{P}(A) = \{X : X \subseteq A\}$
2. Size: $|\mathcal{P}(A)| = 2^{|A|}$
3. If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

MAPPINGS CHEAT SHEET

1. Definition of a Function

- A function $f : A \rightarrow B$ is a rule that assigns **each element of set A** (domain) to **exactly one element of set B** (codomain).
- Domain** = input set
- Codomain** = possible outputs
- Range** = actual outputs used

2. Types of Mappings

(A) Based on Image Coverage

- Injective (One-to-One)**
 - No two elements of A map to the same element of B .
 - $f(x_1) = f(x_2) \implies x_1 = x_2$.
- Surjective (Onto)**
 - Every element of B is mapped by at least one element of A .
 - Range = Codomain.
- Bijjective (One-to-One Correspondence)**
 - Both **injective** and **surjective**.
 - Perfect pairing: each $a \in A$ maps to a unique $b \in B$, covering all of B .
 - Invertible: f^{-1} exists.

(B) Special Types

- Identity function:** $f(x) = x$ for all $x \in A$.
- Constant function:** $f(x) = c$ for all $x \in A$.
- Many-to-one function:** Different inputs can map to the same output.
- Into function:** Not all elements of B are covered (Range $\subset$ Codomain).

3. Counting Functions

Let $|A| = m, |B| = n$.

- Total number of functions** from $A \rightarrow B$:

$$n^m$$

(Each of the m elements in A has n choices.)

- Number of injective functions** ($m \leq n$):

$${}_nP_m = \frac{n!}{(n-m)!}$$

- Number of surjective functions:**

$$n! \cdot S(m, n)$$

where $S(m, n)$ = Stirling number of the 2nd kind = ways to partition m elements into n non-empty subsets.

Alternate formula (Inclusion-Exclusion):

$$\text{Surj}(m, n) = \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m$$

- Number of bijective functions** ($m = n$):

$$n!$$

4. Important Notes

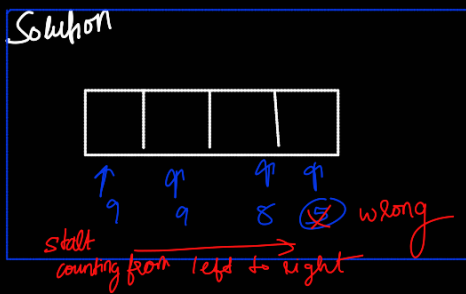
- If $m > n$, **injective functions do not exist**.
- If $m < n$, **surjective functions do not exist**.
- Bijjective $\leftrightarrow m = n$ and all elements are paired uniquely.

5. Graphical Representation

- Function passes the **Vertical Line Test** (each x has exactly one y).
- Injective passes the **Horizontal Line Test** (each y has at most one x).

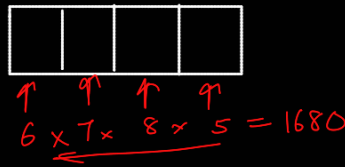
⊕ No of odd. 4 digit nos. without repeating the digits. (First digit cannot be zero, last digit needs to be odd).

Solution

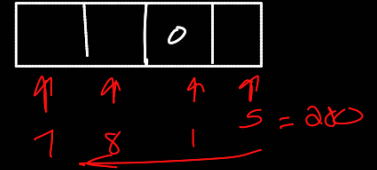
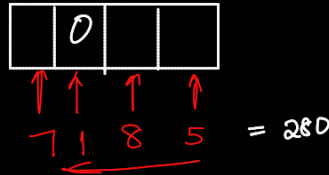


Case by case

1) No zeros in 4 digit numbers

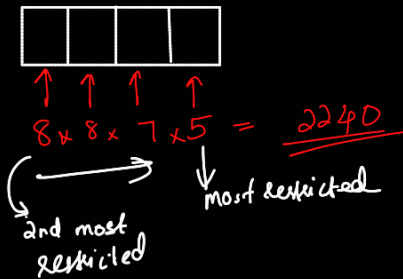


2) Zero in 2nd digit position 3) Zero in 3rd digit

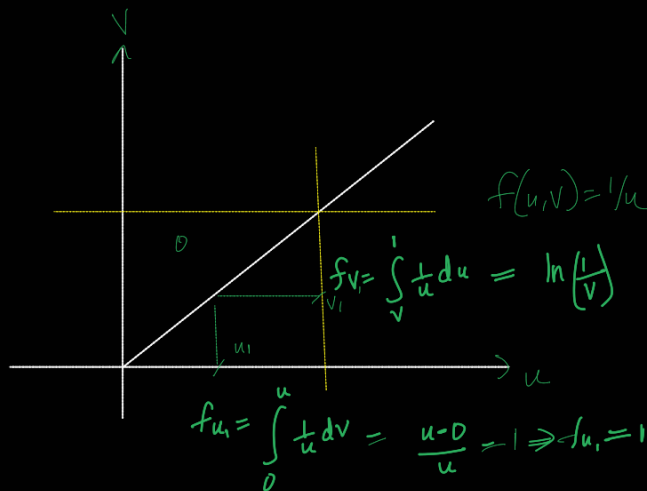
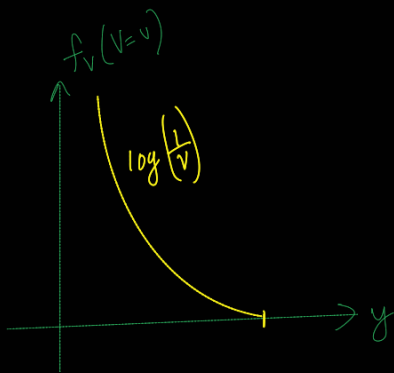


$$\therefore \text{Total} = \underline{2240}$$

Guideline 1: Start counting the most restricted task, especially when applying product rule.



$V \sim \text{uniform}(0,1)$ Find $P(V)$ } marginal
 $V \sim \text{uniform}(0,V)$ $P(V)$ } marginal
 $E(V) \rightarrow$ w/o Adam's law.



$$E(V) = \int_0^1 v \ln\left(\frac{1}{v}\right) dv = \int_1^0 \left[\frac{v^2}{2} \ln v - \frac{v^2}{4} \right] = \frac{1}{4}$$

$$-E(V) = \int_0^1 v \ln v \cdot dv = v \cdot [v \ln v - v] - \int (v \ln v - v) \cdot dv = v^2 \ln v - v^2 - \int v \ln v + \frac{v^2}{2}$$

$$\Rightarrow 2 \int v \ln v \cdot dv = \frac{v^2}{2} \ln v - \frac{v^2}{4} \Rightarrow \int v \ln v = \frac{v^2}{2} \ln v - \frac{v^2}{4}$$

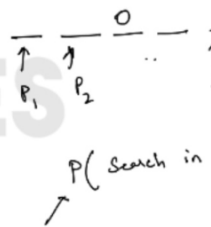
Consider a scenario where a ball is randomly placed in one of n boxes, each with probability P_i for box i . If the ball is in box i , a search of that box will uncover it with probability α_i . The conditional probability that the ball is in box j , given that a search of box i did not uncover it, is:

A. $P(\text{ball in box } j \mid \text{not found in box } i) = \frac{P_j}{1 - \alpha_i P_i}$, if $j \neq i$

B. $P(\text{ball in box } j \mid \text{not found in box } i) = \frac{(1 - \alpha_i) P_j}{1 - \alpha_i P_i}$, if $j \neq i$

C. $P(\text{ball in box } j \mid \text{not found in box } i) = \frac{1 - \alpha_i P_i}{P_j}$, if $j \neq i$

D. $P(\text{ball in box } j \mid \text{not found in box } i) = \frac{P_i}{1 - \alpha_i}$, if $j \neq i$



$$P(B_j | N_i) = \frac{P(N_i | B_j) \cdot P(B_j)}{\sum_{B_j} P(N_i | B_j) \cdot P(B_j)} = \frac{1 \cdot P_j}{\sum_{j \neq i} P_j + P_i (1 - \alpha_i)} = \frac{P_j}{1 - P_i + P_i (1 - \alpha_i)} = \frac{P_j}{1 - \alpha_i P_i}$$

$$\frac{P_j}{1 - P_i + P_i (1 - \alpha_i)} = \frac{P_j}{1 - \alpha_i P_i}$$

$$P(B_j | N_i) = \frac{P_j}{1 - \alpha_i P_i}$$

Let $S = \{1, \dots, n\}$. Suppose that subsets A and B are chosen independently from S so that A and B are equally likely to be any of the 2^n subsets of S (including both the empty set and S itself).

What is the probability that $A \subseteq B$?

$$P(A \subseteq B \mid \|B\| = k) = \frac{2^k}{2^n} = 2^{k-n}$$

$$P(\|B\| = k) = \frac{n C_k}{2^n}$$

$$P(A \subseteq B) = \sum_{k=0}^n \left[P(A \subseteq B \mid \|B\| = k) \cdot P(\|B\| = k) \right] = \sum_{k=0}^n 2^{k-n} \frac{n C_k}{2^n} = \frac{1}{2^n} \sum_{k=0}^n 2^k \cdot n C_k$$

$$P(A \subseteq B) = \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$$



alin_anto 7:54 PM

Intuition wise there is a method without pen and paper.

1) Each element being inside A or B is independent from each other, for n elements.

Let X_i be indicator RV for i th element $\in A$

Let Y_i be indicator RV for i th element $\in B$

Therefore any element i $[X_i, Y_i] \in \{(0,0), (0,1), (1,0), (1,1)\}$

Each element has $3/4$ probability for (not being in A or being in both A & B)

There for A being subset of B for n elements is $(3/4)^n$ (edited)

EXAMPLE 4.5g A secretary has typed N letters along with their respective envelopes. The envelopes get mixed up when they fall on the floor. If the letters are placed in the mixed-up envelopes in a completely random manner (that is, each letter is equally likely to end up in any of the envelopes), what is the expected number of letters that are placed in the correct envelopes?

$A_i =$ Indicator RV of i th card becoming in correct envelope
 $A_1, A_2, \dots, A_n$ are not independent but $E(A_i) = \frac{1}{n} \sim \text{Bern}(\frac{1}{n})$
 $E(A_1 + A_2 + A_3 + \dots + A_n) = n \cdot \frac{1}{n} = 1$



Let $X = X_1 + X_2 + \dots + X_n \rightarrow$ # cards in correct envelope

$$P(A_{k+1} \vee A_{k+2} \vee \dots \vee A_{k+n} \mid A_1, A_2, A_3, \dots, A_k) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots + \frac{1}{(n-k)!} \rightarrow \text{Standard result from derangement's problem} \\ \sim 1 - \frac{1}{e}, n \rightarrow \infty$$

Standard result from derangement's problem & Inclusion exclusion

EXAMPLE 4.5g A secretary has typed N letters along with their respective envelopes. The envelopes get mixed up when they fall on the floor. If the letters are placed in the mixed-up envelopes in a completely random manner (that is, each letter is equally likely to end up in any of the envelopes), what is the expected number of letters that are placed in the correct envelopes?

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$P(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n)$
 Probability (at least one card getting placed)

$1 - P(A_1 \cup A_2 \dots A_n)$
 $P(\text{no card getting placed correctly})$

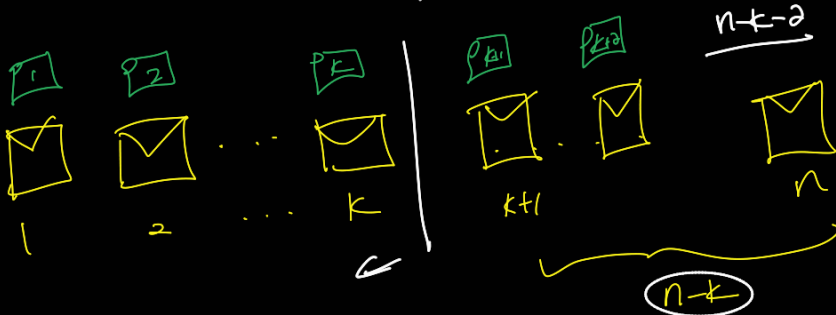
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$$P(A_{k+1} \cup A_{k+2} \cup \dots \cup A_n \mid A_1, A_2, A_3, \dots, A_k) = \left[P(B_{k+1}) + P(B_{k+2}) + \dots + P(B_n) \right] = 1 - \sum_{i < j} P(B_i \cap B_j) + \sum_{i < j < k} P(B_i \cap B_j \cap B_k) - \dots$$

→ given first k cards correctly placed
 Probability of at least one from $k+1$ to n getting correctly placed

$$(-1)^{n-k} P(B_{k+1} \cap B_{k+2} \cap \dots \cap B_n)$$

$P(B_{k+1}) = \frac{P(k+1^{\text{th}} \text{ card going in correct envelope})}{\text{All } k \text{ cards before } k+1 \rightarrow \text{correctly}}$



$$P(B_{k+1}) = \frac{(n-k-1)!}{(n-k)!} = \frac{1}{(n-k)(n-k-1)}$$

$$P(B_{k+1} \cap B_{k+2} \cap \dots \cap B_{k+m}) = \frac{(n-k-m)!}{(n-k)!} = \frac{1}{(n-k) P_m}$$

terms inside summation → $(n-k-m)$

Total all terms

$$P(B_{k+1} \cap B_{k+2} \cap \dots \cap B_{k+m}) = \frac{n-k-m}{n-k P_m} = \frac{1}{m!}$$

$$= P(A_{k+1} \cup A_{k+2} \cup \dots \cup A_n \mid A_1, A_2, A_3, \dots, A_k) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \dots + \frac{1}{(n-k)!}$$

$$1_C = P(A_{k+1} \vee A_{k+2} \vee A_{k+3} \dots A_n \mid \frac{C}{A_1, A_2, A_3 \dots A_k}) = \sum_{i=1}^{n-k} (-1)^{i-1} \frac{1}{i!} = \boxed{1 - \frac{1}{e} \quad n \rightarrow \infty}$$

demonstrates problem

$1-k =$

$$P(\underbrace{\bar{A}_{k+1} \cap \bar{A}_{k+2} \dots \bar{A}_n}_D | C)$$

→ non of the $k+1$ to n cards are correctly placed /
1st k cards are correctly placed.

$$P(D|L) = \frac{1}{0!} - \frac{1+1}{1!} + \frac{1+1}{2!} - \frac{1+1}{3!} + \frac{1}{4!} \dots \frac{1}{n-1!}$$

$$P(q_c) = \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}$$

$$P(C) \cdot P(D|C) = P(C, D)$$

1st k card collected placed

No other cards placed

→ There is nothing special abt fast k cards
→ Prob symmetric as long as k is fixed

$$P(X=k) = nC_k \cdot P(C|D) = nC_k \cdot P(C) \cdot P(L|D)$$

↳ #ways k cards can be fixed

$$P(L) = P(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_k) = \frac{(n-k)!}{n!} = \frac{1}{n P_k}$$

$$P(X=k) = nC_k \cdot \frac{P(L)}{P(L|D) = 1-k} \cdot \left[1 - \underbrace{P(A_{k+1} \vee A_{k+2} \vee \dots \vee A_n / A_1, A_2, \dots, A_k)}_k \right]$$

$$P(X=k) = n C_k \frac{(n-k)!}{n!} \left[1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots - \frac{1}{(n-k)!} \right) \right]$$

$$P(X=k) = \frac{1}{k!} \left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(n-k)!} \right]$$

$$P(X=k) = \frac{1}{k!} \sum_{i=0}^{n-k} (-1)^i \frac{1}{i!}$$

$$E(x) = \sum_{k=1}^n \frac{1}{(k-1)!} \sum_{i=0}^{n-k} (-1)^i \frac{1}{i!}$$

$$E(x) = \sum_{k=1}^n \sum_{i=0}^{n-k} \frac{(-1)^i}{(k-1)!} \cdot \frac{1}{i!}$$

$$E(X) = \sum_{t=0}^{n-1} \sum_{i=0}^{n-1-t} \frac{(-1)^i}{i! \cdot t!}$$

$$E(x) = \sum_{t=0}^{n-1} \sum_{x=t}^{n-1} (-1)^{x-t} \frac{1}{(x)!} \binom{x}{t}$$

$$E(x) = \sum_{x=0}^{n-1} \sum_{t=0}^x (-1)^{x-t} \frac{1}{x!} \binom{x}{t}$$

put $x = i + t$

$$t=0 \quad x \rightarrow 0 \text{ for } n-1$$

$$t = n-1 \quad X \rightarrow n-1$$

$$= \sum_{x=0}^{n-1} \frac{1}{x!} \sum_{t=0}^x (-1)^{x-t} \binom{x}{t} = \begin{cases} \frac{1}{0!} \cdot 1, & x=0 \\ \sum_{x=1}^{n-1} \frac{1}{x!} \cdot \underbrace{(1-1)^x}_{=0}, & x \neq 0 \end{cases}$$

$$\Rightarrow E(k) = 1 //$$

for $n \rightarrow \infty$

$$p(x=k) = \frac{1}{k!} \cdot \frac{1}{e}$$

$$E(x) = \sum_{k=1}^n \frac{1}{(k-1)!} \cdot \frac{1}{e} = \frac{1}{e} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} = \frac{e}{e} = 1 //$$

$$b \leq n-i \quad f(i, j, n) = \sum_{j=m}^{n-a} \sum_{i=a}^{b-j} f(i, j, n)$$

$$① \rightarrow P(A) = 1 - P(A^c)$$

$\rightarrow P(A)$ — which is $P(A^c)$

$P(X)$

$$② \rightarrow P(C) = P(\underbrace{I_1}_{\downarrow P(I_1)} \wedge \underbrace{I_2}_{\downarrow P(I_2)} \wedge I_3)$$

a.g | Indicator-RV
a.b | iid — inde identical
a.c | id — identical

③ Use Inclusion-Exclusion

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

④ Condition of any evidence $\rightarrow$ Is it easier to calculate with evidence

$E(X) \rightarrow$ ① Linearity

② Adam Law

③ Condition expectation

④ MGF

⑤ Joint distribution

⑥ $\sum x P(X=x)$

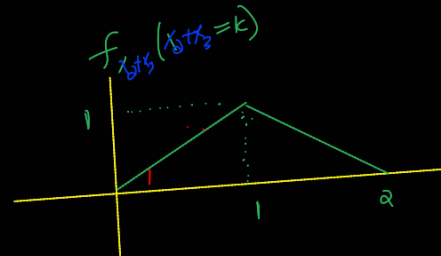
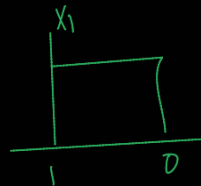
$$\int_0^{\infty} x f(x) dx$$

x_1, x_2, x_3 are iid $\text{uniform}(0,1)$ · $P(x_1 > x_2 + x_3)$?

$$P(x_1 > t) = \int_t^1 P(x_1 = x) dx \Rightarrow P(x_1 > x_2 + x_3) = \int_0^1 \underbrace{f_{x_2+x_3}(x_2+x_3=v)}_{\text{convolution result}} \cdot \underbrace{P(x_1 > v)}_{(1 - \text{marginal CDF})} dv$$

$$P(x_1 > t) = (1-t)$$

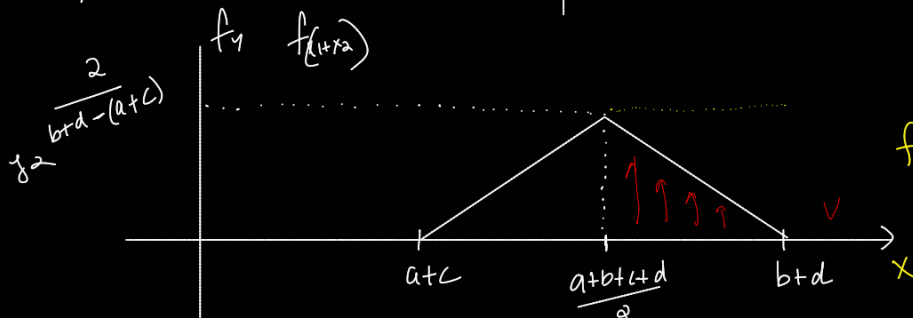
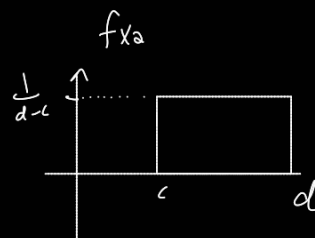
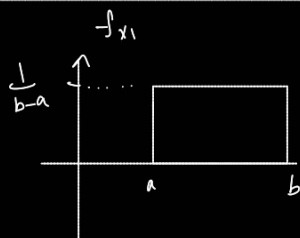
$$= \int_0^1 v(1-v) dv = \left[\frac{v^2}{2} - \frac{v^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$



$$f_{x_2+x_3}(k) = \begin{cases} k & 0 < k < 1 \\ 2-k & 1 < k < 2 \end{cases}$$

$x_1 \sim \text{Uniform}(a,b)$
 $x_2 \sim \text{Uniform}(c,d)$

$$Y = x_1 + x_2$$



$$f_Y(k) = \begin{cases} 0, & k < a+c \\ 0, & k > b+d \\ f_1(y) & a+c \leq k \leq \frac{a+b+c+d}{2} \\ f_2(y) & \frac{a+b+c+d}{2} < k \leq b+d \end{cases}$$

$$f_{1Y} = \frac{4}{[b+d-(a+c)]^2} [x - (a+c)]$$

$$f_{2Y} = \frac{4}{b+d-(a+c)} - f_{1Y}$$

Question

Let (X, Y) be random variables having the joint probability mass function

$$f(x, y) = \begin{cases} \binom{10}{x} \binom{5}{y} \left(\frac{1}{4}\right)^{x-y+5} \left(\frac{3}{4}\right)^{y-x+10}, & x = 0, 1, \dots, 10; y = 0, 1, \dots, 5 \\ 0, & \text{otherwise} \end{cases}$$

Let $Z = Y - X + 10$. Then $E[Z]$ equals

(Answer till two decimal places)

$$f(z) = \sum_{y=0}^5$$

$$Z = Y - X + 10$$

$$f_{X,Y} = \sum_{y=0}^5 \binom{10}{x} \binom{5}{y} \left(\frac{1}{4}\right)^{x-y+5} \left(\frac{3}{4}\right)^{y-x+10}$$

0 Linearity

$$f(x) = \frac{\binom{10}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}}{f_x} \frac{\binom{5}{y} \left(\frac{1}{4}\right)^{5-y} \left(\frac{3}{4}\right)^y}{f_y}$$

$$X \sim \text{Binomial}(10, 1/4) \quad Y \sim \text{Binomial}(5, 3/4)$$

$$E(X) = \frac{10}{4}$$

$$E(Y) = \frac{15}{4}$$

$$E(Z) = \frac{15-10}{4} + 10 = \frac{5}{4} = 1.25$$

A coin comes up heads with probability 0.25 and tails with probability 0.75. The coin is flipped repeatedly until the sequence T, T, H appears for the first time. That is, flipping stops when the most recent three outcomes are T, T, H. Let X be the number of flips required to observe this pattern. Find the expected value $E[X]$.

$$\text{Let } E(X|T) = \mu_T \\ E(X|TT) = \mu_{TT}$$

$$p=0.25 \quad q=0.75 \quad \mu = E(X) = E(X|HH) \cdot p(HH) + E(X|HT) \cdot p(HT) + E(X|TH) \cdot p(TH) + E(X|TT) \cdot p(TT) \rightarrow \text{LTP}$$
$$\mu = (2+\mu) \cdot \frac{1}{16} + (1+\mu_T) \cdot \frac{3}{16} + (2+\mu) \cdot \frac{3}{16} + (\mu_{TT}) \cdot \frac{9}{16} \quad \text{--- (1)}$$

$$E(X|T) = \mu_T = E(X|TT) \cdot \frac{3}{4} + E(X|TH) \cdot \frac{1}{4}$$
$$\mu_T = \mu_{TT} \cdot \frac{3}{4} + (2+\mu) \cdot \frac{1}{4} \quad \text{--- (2)}$$
$$\mu_T = 6 \cdot \frac{3}{4} + \frac{1}{2} + \frac{\mu}{4}$$
$$\mu_T = 5 + \frac{\mu}{4} \quad \text{--- (3)}$$

$$E(X|TT) = E(X|TTT) \cdot \frac{1}{4} + E(X|TTTH) \cdot \frac{3}{4}$$
$$\mu_{TT} = 3 \cdot \frac{1}{4} + (1+\mu_{TT}) \cdot \frac{3}{4}$$
$$\frac{1}{4} \cdot \mu_{TT} = \frac{3}{4} + \frac{3}{4} \Rightarrow \mu_{TT} = 6 \quad \text{--- (4)}$$

$$eq 1 + eq 3 \Rightarrow \mu = (2+\mu) \cdot \frac{1}{16} + \left(1+5+\frac{\mu}{4}\right) \cdot \frac{3}{16} + (2+\mu) \cdot \frac{3}{16} + \frac{6 \times 9}{16}$$
$$\mu = \mu \left[\frac{1}{16} + \frac{3}{4 \times 16} + \frac{3}{16} \right] + \frac{2}{16} + \frac{6 \times 3}{16} + \frac{6}{16} + \frac{6 \times 9}{16}$$
$$\mu = \mu \left[\frac{4+3+12}{64} \right] + \frac{2+18+6+54}{16}$$
$$\mu \cdot \frac{45}{64} = \frac{80}{16} \Rightarrow \mu = \frac{80 \times 64}{16 \times 45} = \frac{64}{9} \approx 7.1$$

$$\frac{64}{9} = 7.1$$

A professor is known to give random grades in every paper to students from the set {A,B and C} arbitrarily, with every grade equally likely. Please calculate expected number of papers to be evaluated for every grade from {A,B and C} are picked?

(Answer till one decimal place)

$X = \# \text{ papers evaluated so that every grade is picked}$
 $X = 1 + X_1 + X_2$
 $X_1 = \# \text{ papers evaluated to get 2 distinct papers } G_1 \neq G_2 \mid G_1$
 $X_2 = \# \text{ More papers to be evaluated to get 3rd distinct paper } G_3 \mid G_1, G_2$
 $E(X) = 1 + E(X_1) + E(X_2)$
 $\quad = 1 + 3/2 + 3 = 5.5$
 $X_1 \sim \text{geometric}(2/3)$
 $E(X_1) = 3/2 = 1.5$
 $X_2 \sim \text{geometric}(1/3)$
 $E(X_2) = 3$

A fair dice (with faces numbered $1, \dots, 6$) is independently rolled repeatedly. Let X denote the number of rolls till an even number is seen and let Y denote the number of rolls till 3 is seen. Evaluate $E(Y \mid X = 2)$.

An urn contains three balls, red, blue and green. Balls are chosen uniformly at random with replacement. Calculate expected number of different colors of balls obtained before obtaining first red ball? (Don't count red color)

E.g. sequence: green, blue, green, red (The number of different colors obtained 2, because blue and green both observed)

Sequence 2: green, green, red. (The number of different colors obtained 1, because only green is observed)

$X = \# \text{ Distinct colors before red}$
 $X = I_g + I_b$
 $E(X) = E(I_g) + E(I_b)$
 $E(X) = 2E(I_g)$
 $E(X) = 1$

$I_g = \text{Indicator of green appearing before Red}$
 $I_b = \text{Indicator of blue " before Red}$
 $I_g \text{ \& } I_b \text{ not independent but identical}$
 $E(I_g) = P(I_g = 1) = 1/2$
 Since Probability of green appearing before red
 $= \text{Probability of green not appearing}$

$R \mid \begin{array}{l} 1/3 \\ BR \mid 1/3 \cdot 1/3 \\ BBR \mid 1/3 \cdot 1/3 \cdot 1/3 \\ BBBR \mid 1/3 \cdot 1/3 \cdot 1/3 \cdot 1/3 \end{array}$
 $= 1/3 + 1/3^2 + 1/3^3 + \dots + 1/3^n \quad n \rightarrow \infty$
 $= \frac{1/3}{1 - 1/3} = \frac{1/3}{2/3} = \underline{\underline{1/2}}$

Generalisation $n \rightarrow$ distinct non red balls & 1 red ball

$X = I_1 + I_2 + I_3 + \dots + I_n$
 $E(X) = n E(I_1)$

$E(I_1) = P(I_1 = 1) = 1 - P(I_1 = 0)$
 $\quad = 1/2$

$E(X) = \underline{\underline{n/2}}$

$I_1, I_2, \dots, I_n$ dependent but identical

$I_i \rightarrow$ indicator for i^{th} color ball appearing before red

$P(I_1 = 0)$
 $R \rightarrow 1/(n+1)$
 $XR \rightarrow \frac{n-1}{n+1} \cdot \frac{1}{n+1}$
 $XXR \rightarrow \frac{(n-1)^2}{(n+1)^2} \cdot \frac{1}{n+1}$
 $\vdots$
 $\left. \begin{array}{l} R \\ XR \\ XXR \\ \vdots \end{array} \right\} = \frac{1}{n+1} \left[\frac{(n-1)^0}{(n+1)^0} + \frac{(n-1)^1}{(n+1)^1} + \frac{(n-1)^2}{(n+1)^2} + \dots + \frac{(n-1)^{\infty}}{(n+1)^{\infty}} \right]$
 $= \frac{1}{n+1} \frac{n+1}{(n+1) - (n-1)} = \underline{\underline{1/2}}$

A fair dice (with faces numbered 1, ..., 6) is independently rolled repeatedly. Let X denote the number of rolls till an even number is seen and let Y denote the number of rolls till 3 is seen. Evaluate $E(Y | X = 2)$.

— Even —

$$E(Y|X=2) = E(Y|X=2, Y<2) \cdot P(Y<2) + E(Y|X=2, Y \geq 2) \cdot P(Y \geq 2)$$

$$\mu = 1 \cdot (1/3) + 2 \cdot (2/3) = 5/3$$

$$\mu = 1/3 + 16/3 = 17/3$$

= 2 + 2
2 ~ geometric (1/6)

The time it takes a central processing unit (CPU) to process a certain type of job is normally distributed. A term $\hat{\sigma}_1^2$ is defined as:

$$\hat{\sigma}_1^2 = \frac{1}{15} \sum_{n=1}^{15} x_n^2$$

where x_n represents the observed processing times for each job in a sample of 15 jobs, and the sample mean is 0.

The expected value of $\hat{\sigma}_1^2$ is known to be $\frac{42}{5}$.

What is the probability that the variance of this sample will exceed 12?

$$s^2 = \text{Sample variance} = \frac{1}{14} \sum_{n=1}^{15} (x_n - \bar{x}_n)^2 = \frac{1}{14} \sum_{n=1}^{15} (x_n)^2 = \frac{15}{14} \hat{\sigma}_1^2$$

$$E(s^2) = \frac{15}{14} E(\hat{\sigma}_1^2) = \frac{15}{14} \times \frac{42}{5} = \frac{3 \times 6}{2} = 9 \Rightarrow \sigma^2 = 9$$

① Expectation of Sample Variance = population variance (σ^2)

$$P(s^2 > 12) = P\left(\frac{s^2(n-1)}{\sigma^2} > \frac{12(n-1)}{\sigma^2}\right) \Rightarrow P(\chi^{n-1} > \frac{12 \times 14}{9}) \rightarrow \text{from table}$$

② if $s^2 = \text{Sample variance}$ then $\frac{s^2(n-1)}{\sigma^2} \sim \chi^{n-1}$

$$\frac{X - \mu}{\sigma} = Z$$

$$X = \sigma Z + \mu$$

$$X^2 = \sigma^2 Z^2 + \mu^2 + 2\sigma\mu Z$$

$$E(X^2) = \sigma^2 + \mu^2$$

$$\frac{1}{15} \sum (x_i^2) = \sigma^2 + \mu^2$$

The Chi-Square Statistic for Sample Variance

The formula that transforms a sample variance (s^2) into a variable that follows a chi-square (χ^2) distribution is:

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

where:

- n is the sample size.
- s^2 is the sample variance.
- σ^2 is the true population variance (a known or hypothesized value).
- The resulting χ^2 statistic has $\nu = n - 1$ degrees of freedom.

Conditions and Rationale

- Assumption of Normality:** The underlying population from which the sample is drawn must be normally distributed. This is a strict requirement for this relationship to be valid.
- Degrees of Freedom:** The use of $n - 1$ (degrees of freedom) rather than n in the numerator accounts for the fact that the sample mean ($\bar{x}$) is used to calculate the sample variance. Calculating deviations around the sample mean, instead of the true population mean, causes the sample variance to slightly under-estimate the population variance, and the $n - 1$ correction addresses this bias.
- Nature of the Distribution:** The value is always positive, consistent with the properties of both variance and the chi-square distribution. The chi-square distribution itself is defined as the sum of the squares of independent standard normal random variables.

A biased coin is flipped repeatedly. Let N denote the number of flips required until two of the most recent three flips are heads. (For example, if the first two flips are heads, then $N = 2$.) Compute the expected value of N , given that the probability of heads is 0.5.

$$p = 0.5$$

HTT	— 1/8
HTH	— 1/8
THT	— 1/8
HH	— 1/4
	$\Sigma = 1$

$$\mu = 1/2 \cdot [1 + \mu] + \frac{1}{4} \times 2 + \frac{1}{8} \times (3 + \mu) + \frac{1}{8} \times 3$$

$$\mu = \mu \left[\frac{1}{2} + \frac{1}{8} \right] + \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{3}{8}$$

$$\mu \left[\frac{8 - 4 - 1}{8} \right] = \frac{4 + 4 + 3 + 3}{8}$$

$$\mu = \frac{14}{3} = \underline{\underline{4.66}}$$

Let a_1, a_2, a_3, a_4, a_5 be independent random variables such that

$$a_i \sim \text{Uniform}(0, i), \quad i = 1, 2, 3, 4, 5.$$

What is the probability that

$$X_1 < X_2$$

$$a_1 < a_2 < a_3 < a_4 < a_5$$

$$a_1 \sim X_1$$

$$P(a_5 > a_4 > a_3 > a_2 > a_1) = \int_0^1 \int_0^2 \int_0^3 \int_0^4 \int_0^5 \frac{1}{5!} da_5 \cdot da_4 \cdot da_3 \cdot da_2 \cdot da_1 = \frac{1}{5!} \int_0^1 \int_0^2 \int_0^3 \int_0^4 (5 - a_4) da_4 da_3 da_2 da_1 = \frac{1}{5!} \int_0^1 \int_0^2 \int_0^3 5(4 - a_3) - \frac{1}{2} a_3^2 da_3 da_2 da_1$$

$$= \frac{1}{5!} \int_0^1 \int_0^2 \left[12(3 - a_2) - \frac{5}{2} (9 - a_2^2) + \frac{1}{6} (3^3 - a_2^3) \right] da_2 da_1 = \frac{1}{5!} \int_0^1 \left[18(2 - a_1) - 6(4 - a_1^2) + \frac{5}{2 \times 3} (8 - a_1^3) - \frac{16 - a_1^4}{2 \times 3 \times 4} \right] da_1$$

$$= \frac{18 - 9 + 2 - \frac{25}{24 \times 5} + \frac{1}{5 \times 24}}{5!} = \frac{11 - 1/5}{5!} = \frac{54}{5 \times 5!} = \underline{\underline{0.09}}$$

1. Consider the Monty Hall problem, except that Monty enjoys opening Door 2 more than he enjoys opening Door 3, and if he has a choice between opening these two doors, he opens Door 2 with probability p , where $\frac{1}{2} \leq p \leq 1$.

To recap: there are three doors, behind one of which there is a car (which you want), and behind the other two of which there are goats (which you don't want). Initially, all possibilities are equally likely for where the car is. You choose a door, which for concreteness we assume is Door 1. Monty Hall then opens a door to reveal a goat, and offers you the option of switching. Assume that Monty Hall knows which door has the car, will always open a goat door and offer the option of switching, and as above assume that if Monty Hall has a choice between opening Door 2 and Door 3, he chooses Door 2 with probability p (with $\frac{1}{2} \leq p \leq 1$).

- (a) Find the unconditional probability that the strategy of always switching succeeds (unconditional in the sense that we do not condition on which of Doors 2,3 Monty opens).

- (b) Find the probability that the strategy of always switching succeeds, given that Monty opens Door 2.

$$P(C|S) = \text{still the same since we don't know which door opened}$$

$$P(C|S, D_2) = P(C|S, D_1, D_2) \cdot P(D_1|D_2) + P(C|S, D_2, D_3) \cdot P(D_3|D_2)$$

$$\therefore P(C|S, D_2) = P(D_3|D_2) = \frac{P(D_2|D_3) \cdot P(D_3)}{P(D_2|D_3) \cdot P(D_3) + P(D_2|D_1) \cdot P(D_1)} = \frac{1 \cdot (1/3)}{1 \cdot (1/3) + p \cdot (1/3)} = \boxed{\frac{1}{1+p}}$$

C = Event $\hookrightarrow$ Win the car
 S = Event $\hookrightarrow$ We switched
 D_j = Event $\hookrightarrow$ Opened door j
 D_j = Event $\hookrightarrow$ Car behind door j
 we need $P(C|S)$

LOTP

$$P(C|S) = P(C|S, D_1) \cdot P(D_1) + P(C|S, D_2) \cdot P(D_2) + P(C|S, D_3) \cdot P(D_3)$$

$$P(C|S) = 0 \cdot (1/3) + 1 \cdot (1/3) + 1 \cdot (1/3)$$

$$P(C|S) = \boxed{2/3}$$

1. (a) Consider the following 7-door version of the Monty Hall problem. There are 7 doors, behind one of which there is a car (which you want), and behind the rest of which there are goats (which you don't want). Initially, all possibilities are equally likely for where the car is. You choose a door. Monty Hall then opens 3 goat doors, and offers you the option of switching to any of the remaining 3 doors.

Assume that Monty Hall knows which door has the car, will always open 3 goat doors and offer the option of switching, and that Monty chooses with equal probabilities from all his choices of which goat doors to open. Should you switch? What is your probability of success if you switch to one of the remaining 3 doors?

- (b) Generalize the above to a Monty Hall problem where there are $n \geq 3$ doors, of which Monty opens m goat doors, with $1 \leq m \leq n-2$.

since we have chosen door 1
 $P(C|S, D_1) = 0$; ie we won't win the car, if we switch when the car is behind Door 1

$$\text{therefore} \rightarrow P(C|S) = \sum_{j=2}^n [P(C|S, D_j) \cdot P(D_j)]$$

$$P(D_j) = \frac{1}{n} \text{ } \} \text{ unconditionally}$$

therefore

$$P(C|S) = \frac{1}{n} \sum_{j=2}^n [P(C|S, D_j)]$$

Since Monty opens m doors,
 there are $n-m-1$ doors remaining to switch to. Probability of us choosing the correct door among this is therefore

$$\frac{1}{n-m-1}$$

$$\text{therefore} \rightarrow P(C|S) = \frac{1}{n} (n-1) \cdot \frac{1}{n-m-1}$$

$$P(C|S) = \frac{n-1}{n(n-m-1)}$$

$$n \geq 3 \\ 1 \leq m \leq n-2$$

Monty hall case,

$$n=3 \\ m=1 \quad P(C|S) = \frac{2}{3(3-1-1)} = \frac{2}{3 \cdot 1} = 2/3$$

Our case,

$$n=7 \\ m=3 \quad P(C|S) = \frac{6}{7(7-3-1)} = \frac{6}{7 \cdot 3} = \frac{6}{21} = \frac{2}{7}$$

2. The odds of an event with probability p are defined to be $\frac{p}{1-p}$, e.g., an event with probability $3/4$ is said to have odds of 3 to 1 in favor (or 1 to 3 against). We are interested in a hypothesis H (which we think of as an event), and we gather new data as evidence (expressed as an event D) to study the hypothesis. The *prior* probability of H is our probability for H being true before we gather the new data; the *posterior* probability of H is our probability for it after we gather the new data. The *likelihood ratio* is defined as $\frac{P(D|H)}{P(D|H^c)}$. **likelihood ratio**

(a) Show that Bayes' rule can be expressed in terms of odds as follows: *the posterior odds of a hypothesis H are the prior odds of H times the likelihood ratio.*

(b) As in the example from class, suppose that a patient tests positive for a disease afflicting 1% of the population. For a patient who has the disease, there is a 95% chance of testing positive (in medical statistics, this is called the *sensitivity* of the test); for a patient who doesn't have the disease, there is a 95% chance of testing negative test (in medical statistics, this is called the *specificity* of the test).

The patient gets a second, independent test done (with the same sensitivity and specificity), and again tests positive. Use the odds form of Bayes' rule to find the probability that the patient has the disease, given the evidence, in two ways: in one step, conditioning on both test results simultaneously, and in two steps, first updating the probabilities based on the first test result, and then updating again based on the second test result.

H = Event {patient has the disease}

D = Event {patient tests positive}

$$\left. \begin{array}{l} \text{sensitivity} = P(D|H) \\ \text{specificity} = P(D^c|H^c) \end{array} \right\} \Rightarrow \text{likelihood} = \frac{\text{sensitivity}}{1 - \text{specificity}}$$

$$\text{In our case likelihood} = \frac{0.95}{0.05} = 19$$

$\Rightarrow$ After first test odds increase by 19
After second test odds increase by 19^2

$$P(H|D) = \frac{P(D|H) \cdot P(H)}{P(D|H) \cdot P(H) + P(D|H^c) \cdot P(H^c)}$$

Let the posterior probability $P(H|D) = p'$
 $\rightarrow$ posterior odds = $\frac{p'}{1-p'}$

$$= \frac{P(D|H) \cdot P(H)}{P(D|H) \cdot P(H) + P(D|H^c) \cdot P(H^c)}$$

$$= \frac{P(D|H) \cdot P(H)}{P(D|H^c) \cdot P(H^c)} = \frac{P(D|H)}{P(D|H^c)} \cdot \frac{P}{1-P}$$

$$\frac{p'}{1-p'} = \frac{P(D|H)}{P(D|H^c)} \cdot \frac{P}{1-P}$$

4. Calvin and Hobbes play a match consisting of a series of games, where Calvin has probability p of winning each game (independently). They play with a "win by two" rule: the first player to win two games more than his opponent wins the match. Find the probability that Calvin wins the match (in terms of p), in two different ways:

(a) by conditioning, using the law of total probability.

(b) by interpreting the problem as a gambler's ruin problem.

C = Event {Calvin wins the match}

a) Using LTTP

$$X = \text{Random variable} \sim \text{Bin}(2, p)$$

of wins for player A in the first 2 games

$$C = \text{Event \{Player A wins\}}$$

$$P(C) = P(C|X=0) \cdot 2p^2 + P(C|X=1) \cdot 2p^2 + P(C|X=2) \cdot p^2$$

$\rightarrow = 0$ $\rightarrow = 1$ $\rightarrow = 2$

$$P(C) = P(C) \cdot 2p^2 + p^2$$

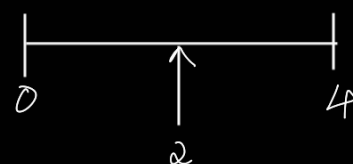
$$P(C) = \frac{p^2}{1-2p^2} = \frac{p^2}{p^2 + q^2}$$

$$\begin{aligned} 1 - 2p^2 &= 1 + 2p(p-1) \\ &= 1 + 2p^2 - 2p \\ &= p^2 + p^2 - 2p + 1 \\ &= p^2 + (1-p)^2 \\ 1 - 2p^2 &= p^2 + q^2 \end{aligned}$$

Interpreting the previous problem as gambler's ruin.

$$P_2 = \frac{1 - \left(\frac{q}{p}\right)^2}{1 - \left(\frac{q}{p}\right)^4} = \frac{p^4 - p^2 q^2}{p^4 - q^4} = \frac{p^2 (\cancel{p^2} - \cancel{q^2})}{(\cancel{p^2} - \cancel{q^2})(p^2 + q^2)} = \frac{p^2}{p^2 + q^2}$$

$$P_2 = \frac{p^2}{p^2 + q^2}$$



here $n=4$
 $x=2$

Q8

Let (X, Y) be a random vector such that, for any $y > 0$, the conditional probability density function of X given $Y = y$ is:

$$f_{X|Y=y}(x) = ye^{-yx}, \quad x > 0.$$

The marginal probability density function of Y is:

$$g(y) = ye^{-y}, \quad y > 0.$$

Find $\mathbb{E}[Y|X = 1]$ (correct up to one decimal place).

$$f(x, y) = f_{X|Y=y} \cdot f_Y(y) = y^2 e^{-y(x+1)} \quad \begin{matrix} x > 0 \\ y > 0 \end{matrix}$$

$$f_{X=1} = \int_0^{\infty} y^2 e^{-2y} dy = \left[y^2 \cdot \frac{e^{-2y}}{-2} - 2y \cdot \frac{e^{-2y}}{4} + 2 \cdot \frac{e^{-2y}}{-8} \right]_0^{\infty} = \frac{2}{8} = 1/4$$

$$E(Y|X) = \int_0^{\infty} y \cdot f_{Y|X=1} dy \quad f_{Y|X=1} = \frac{f_{X=1, Y=y}}{f_{X=1}}$$

$$E(Y|X) = 4 \int_0^{\infty} y \cdot y^2 e^{-2y} dy = 4 \left[y^3 \frac{e^{-2y}}{-2} - 3y^2 \frac{e^{-2y}}{4} + 6y \frac{e^{-2y}}{-8} - \frac{6e^{-2y}}{16} \right]_0^{\infty} = \frac{6 \times 4}{16}$$

$$E(Y|X) = \underline{\underline{3/2 = 1.5}}$$

shortcut

exponential pdf

$$f_{X=x} =$$

$$de^{-dx}$$

$$\int_0^{\infty} y^2 e^{-2y} dy = \frac{1}{2} E(X^2) \quad X \sim \text{expo}(2)$$

$$= \frac{1}{2} \cdot \frac{2!}{2^2} = \frac{1}{4}$$

$$E[X^n] = \frac{n!}{\lambda^n}$$

$$4 \int_0^{\infty} y^3 e^{-2y} dy = 2 \times E(X^3) \quad X \sim \text{expo}(2)$$

$$= 2 \times \frac{3!}{2^3} = \frac{6}{4} = \underline{\underline{1.5}}$$

shortcut using gamma

$$\int_0^{\infty} x^a e^{-x} \frac{dx}{x} = \Gamma(a) = (a-1)!$$

$$\int_0^{\infty} y^2 e^{-2y} dy \quad \left| \begin{array}{l} \text{let } 2y = x \\ dy = \frac{dx}{2} \\ y^2 = \frac{x^2}{4} \end{array} \right.$$

$$\frac{1}{8} \int_0^{\infty} x^3 e^{-x} \frac{dx}{x} = \frac{\Gamma(3)}{8} = \frac{2!}{8} = \frac{2}{8} = \frac{1}{4}$$

$$4 \int_0^{\infty} y^3 e^{-2y} dy \quad \left| \begin{array}{l} 2y = x \\ dy = \frac{dx}{2} \\ y^3 = \frac{x^3}{8} \end{array} \right.$$

$$\frac{4}{8 \times 2} \int_0^{\infty} x^3 e^{-x} \frac{dx}{x} = \frac{\Gamma(4)}{4} = \frac{3!}{4} = \frac{6}{4} = \underline{\underline{1.5}}$$

$$X \sim \text{Gamma}(a, \lambda) \Rightarrow f_X = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x} \quad x > 0$$

$$\Gamma(a) = (a-1)! \quad \int_0^{\infty} f_X = 1 \Rightarrow \int_0^{\infty} x^{a-1} e^{-\lambda x} dx = \frac{\Gamma(a)}{\lambda^a}$$

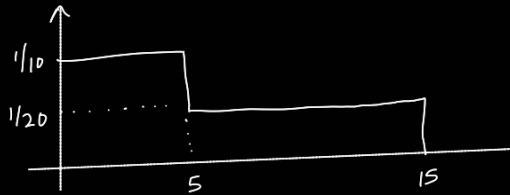
$$\begin{array}{l} dx = y \\ du = \frac{dy}{\lambda} \\ \lambda^{a-1} x^{a-1} = y^{a-1} \end{array}$$

The metro train arrives at the station near your home every quarter hour starting at 6:00 AM. You walk into the station every morning between 7:10 and 7:30 AM, with the time in this interval being a uniform random variable. What is the PDF of the time you have to wait for the first train to arrive?

$$f_X(x) = f_X(x | \text{Reach before 7:15}) \cdot P(\text{Reach before 7:15}) + f_X(x | \text{Reach after 7:15}) \cdot P(\text{Reach after 7:15})$$

$$= \begin{cases} \frac{1}{5} \cdot \frac{5}{20} & 0 < x < 5 \\ 0 & x > 5 \end{cases} + \begin{cases} \frac{1}{20} & 0 < x < 5 \\ \frac{1}{20} & 5 < x < 15 \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{10} & 0 < x < 5 \\ \frac{1}{20} & 5 < x < 15 \end{cases}$$



Q7

Let the random vector (X, Y) have the joint distribution function:

$$F(x, y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0, \\ 1 - e^{-x}, & \text{if } x \geq 0 \text{ and } 0 \leq y < 1, \\ \frac{1-e^{-x}}{4}, & \text{if } x \geq 0 \text{ and } y \geq 1. \end{cases}$$

Let $\text{Var}(X)$ and $\text{Var}(Y)$ denote the variances of the random variables X and Y , respectively. Compute:

$$16 \text{Var}(X) + 32 \text{Var}(Y). \quad \underline{\underline{22}}$$

$$f(x,y) = \begin{cases} 0 \\ e^{-x} \end{cases} \quad x \geq 0, 0 \leq y \leq 1$$

$$f(x,y) = \begin{cases} e^{-x} & x \geq 0, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$F(x,y) = F(x)$
 $AM \cdot e^{-x} \cdot \frac{1-e^{-x}}{1}$
 $1 - e^{-x}$
 $16 + 6 = 22$

$$16 + 6 = \underline{\underline{22}}$$

22

$$E(x) = p$$

$$E(y^2) = p$$

$$\sum y^2 f(y) =$$

$$\binom{2}{1} \cdot 3/4 =$$

$$f(y) = \begin{cases} 1/4 & y=0 \\ 3/4 & y=1 \end{cases}$$

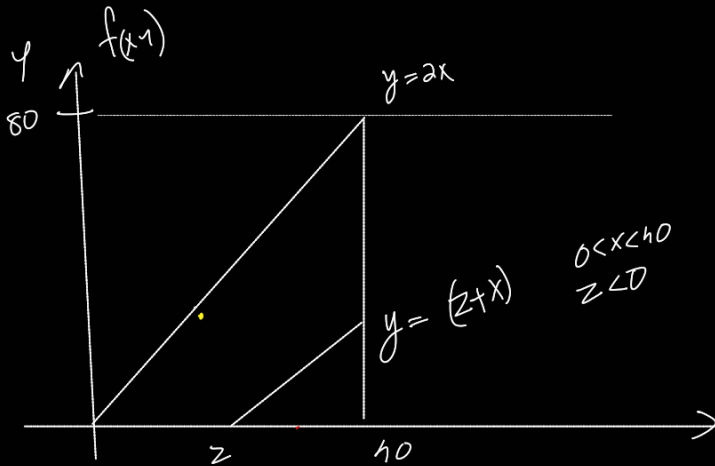
pg 3/16

One of two wheels of fortune, A and B , is selected by the flip of a fair coin, and the wheel chosen is spun once to determine the experimental value of random variable X . Random variable Y , the reading obtained with wheel A , and random variable W , the reading obtained with wheel B , are described by the PDFs

$$f_Y(y) = \begin{cases} 1, & 0 < y \leq 1; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_W(w) = \begin{cases} 3, & 0 < w \leq \frac{1}{3}; \\ 0, & \text{otherwise.} \end{cases}$$

If we are told the experimental value of X was less than $\frac{1}{4}$, what is the conditional probability that wheel A was the one selected?

$$P(A|X < 1/4) = \frac{P(A) \cdot P(X < 1/4|A)}{P(A) \cdot P(X < 1/4|A) + P(B) \cdot P(X < 1/4|B)} = \frac{1/2 \cdot 1/4}{1/2 \cdot 1/4 + 1/2 \cdot 3/4} = \frac{1}{1+3} = \underline{\underline{1/4}}$$



$$y=2x$$

$$f_{XY}(x,y) = \frac{1}{1600} \quad \begin{matrix} 0 \leq x \leq 40 \\ 0 \leq y \leq 80 \end{matrix}$$

$$P(Z=z) \int_z^{40} \frac{1}{1600}$$

$$(f_{z,y})$$

Discrete uniform distribution Uniform $[a, b]$

PDF

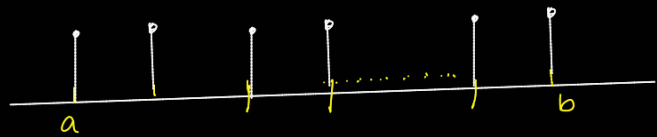
$$P(X=k) = \frac{1}{b-a+1}$$

$$P(X \leq k) = \begin{cases} \frac{k-a+1}{b-a+1}, & a \leq k \leq b \\ 0, & k < a \\ 1, & k > b \end{cases}$$

$$E(X) = \sum_{i=a}^b i \cdot \frac{1}{b-a+1} = \frac{a+b}{2} \cdot \frac{b-a+1}{2} = \frac{a+b}{2}$$

$$E(X^2) = \sum_{i=a}^b i^2 \cdot \frac{1}{b-a+1} =$$

$$Var(X) = \frac{N^2-1}{12} \quad N = \# \text{ element} = b-a+1$$



Use the identity

$$\sum_{k=1}^m k^2 = \frac{m(m+1)(2m+1)}{6}$$

$$\sum_{x=A}^B x^2 = \sum_{k=1}^B k^2 - \sum_{k=1}^{A-1} k^2$$

$$= \frac{B(B+1)(2B+1)}{6} - \frac{(A-1)A(2A-1)}{6}$$

Divide by $n = B - A + 1$:

$$E[X^2] = \frac{B(B+1)(2B+1) - (A-1)A(2A-1)}{6(B-A+1)}$$

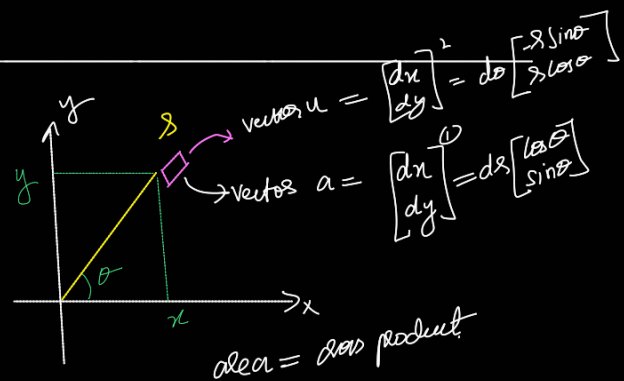
$$s^2 = x^2 + y^2 \rightarrow ds = dx + dy$$

$$\theta = \tan^{-1} y/x \rightarrow d\theta = \frac{1}{1+y^2/x^2} \cdot \frac{dy}{dx}$$

$$y = x \tan \theta$$

$$d\theta = \frac{x^2}{x^2 + y^2} \frac{dy}{dx}$$

$$d\theta = \frac{x^2}{s^2} \frac{dy}{dx}$$



$$x = s \cos \theta$$

$$y = s \sin \theta$$

$$dx = \cos \theta ds - s \sin \theta d\theta$$

$$dy = \sin \theta ds + s \cos \theta d\theta$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \cos \theta & -s \sin \theta \\ \sin \theta & s \cos \theta \end{bmatrix} \begin{bmatrix} ds \\ d\theta \end{bmatrix}$$

$$J(x, \theta) ds d\theta$$

$$dx dy =$$

#emails in time t is given by $N \sim \text{Poisson}(t)$

Q) Find the P.D.F of T , where T is the time until the first email

$$P(N=0) = \frac{e^{-\lambda t} \lambda^0}{0!} = e^{-\lambda t} = \text{probability of not getting any mail in time } t.$$

$$P(T \geq t) = \text{probability of waiting atleast time } t' \text{ until first mail}$$

$$= \text{probability of not getting email in time } t'$$

$$= e^{-\lambda t'}$$

$$CDF = 1 - P(T \geq t) = 1 - e^{-\lambda t'}$$

$$P.D.F f_T(t) = \lambda e^{-\lambda t}$$

Let (X, Y) have a bivariate normal distribution with the joint probability density function:

$$f_{X,Y}(x,y) = \frac{1}{\pi} e^{\left(\frac{3}{2}xy - \frac{25}{32}x^2 - 2y^2\right)}, \quad -\infty < x, y < \infty.$$

Compute $8E[XY]$.

given

$$\frac{1}{\pi} \exp\left(-\frac{1}{2}\left[-3xy + \frac{25}{16}x^2 + 4y^2\right]\right) = \frac{e^{-\frac{1}{2}\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{25}{16} & -3/2 \\ -3/2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}}{\pi}$$

$$2|\Sigma|^{1/2} = 1$$

$$|\Sigma|^{1/2} = 1/2$$

$$|\Sigma| = 1/4$$

$$\Sigma^{-1} = \begin{bmatrix} \frac{25}{16} & -3/2 \\ -3/2 & 4 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 4 & 3/2 \\ 3/2 & 25/16 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1 & 3/8 \\ 3/8 & 25/64 \end{bmatrix}$$

$$E(XY) = \text{Cov}(X, Y) + E(X)E(Y) = \underline{\underline{3/8}}$$

$R|\bar{A}BC$
↓

B+ tree on B attribute

A	B	C
1	7	2
2	7	5
	7	8

unclustered

VONK

1	8	2
2	7	4
3	7	2

B1

4 4 1

B2

5 7 8

6 7 2

L-1

clustered VONK



B# B# B# B# B#

1 Block

block array

Secondary index VONKA

Random variables X and Y are distributed according to the joint PDF

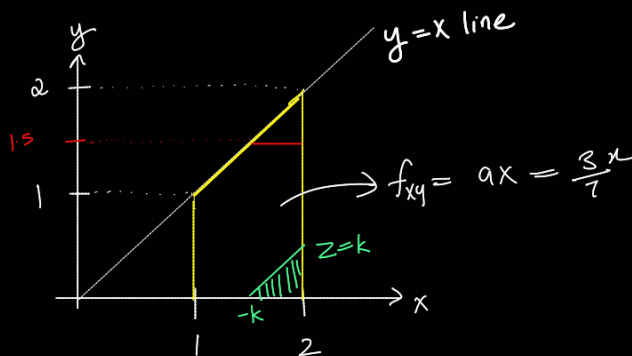
$$f_{X,Y}(x,y) = \begin{cases} ax, & \text{if } 1 \leq x \leq 2 \text{ and } 0 \leq y \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Evaluate the constant a .

(b) Determine the marginal PDF $f_Y(y)$.

(c) Determine the conditional expectation of $1/X$ given that $Y = 3/2$.

(d) Random variable Z is defined by $Z = Y - X$. Determine the PDF $f_Z(z)$.



$$a) 1 = \int_1^2 \int_0^x ax \, dy \, dx = \int_1^2 ax[y]_0^x \, dx = \int_1^2 ax^2 \, dx$$

$$\Rightarrow 1 = a \left[\frac{x^3}{3} \right]_1^2 \Rightarrow 1 = \frac{a}{3} [8 - 1] \Rightarrow a = 3/7$$

$$b) f_Y(y) = \begin{cases} \int_1^2 ax \, dx = \frac{a}{2} [4 - 1] = \frac{3a}{2} = \frac{9}{14} & y \leq 1 \\ \int_y^2 ax \, dx = \frac{a}{2} [4 - y^2] = \frac{6}{7} - \frac{3y^2}{14} & 1 < y < 2 \end{cases}$$

$$f_X(x) = \int_0^x ax \, dy = ax^2 = \frac{3x^2}{7} \quad 1 \leq x \leq 2$$

$$F_Z(Z < k) = \int_1^2 \int_0^{x+k} ax \, dy \, dx = \int_1^2 ax^2 + akx \, dx = \frac{a}{3} (8 - 1) + \frac{ak}{2} (k + 1) = 1 + \frac{9k}{14}$$

To find pdf $\rightarrow$ derivative

$$f_Z(z) = \begin{cases} -\frac{3z^2}{14} + \frac{6}{7} & -2 \leq z \leq -1 \\ \frac{9}{14} & -1 \leq z \leq 1 \end{cases}$$

$$f_{X,Y} = \begin{cases} ax & 1 \leq x \leq 2, 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

$$Z = Y - X \quad f_Z(z) = \int_1^2 f_{X,Y}(x, y/x) f_X(x) \, dx \quad 1 \leq x \leq 2$$

$$f_Z(z) = \int_1^2 ax \, dx$$

$$c) E(1/k | Y = 3/2)$$

$$= \int_{3/2}^2 \frac{ax}{x} \, dx = a [2 - 3/2] = \frac{a}{2} = \frac{3}{14}$$

$$d) Z \text{ range is } [-2, 0]$$

$$F_Z(Z < k) = \int_{-k}^2 \int_0^{x+k} ax \, dy \, dx \quad -2 \leq k \leq -1$$

$$= \int_{-k}^2 ax[x+k] \, dx = \frac{a}{3} [x^3]_{-k}^2 + \frac{ak}{2} [x^2]_{-k}^2$$

$$= \frac{1}{7} [8 + k^3] + \frac{ak}{2} [4 - k^2]$$

$$= \frac{8}{7} + \frac{k^3}{7} + \frac{6k}{7} - \frac{3k^3}{14}$$

$$= -\frac{k^3}{14} + \frac{6k}{7} + \frac{8}{7} \quad -2 \leq k \leq -1$$

$$P(X < Y) = \sum_{k=0}^{\infty} P(X < Y / Y=k) \cdot P(Y=k) = \sum_{y=0}^{\infty} \sum_{x=0}^{y-1} 10C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x} \cdot \frac{11C_y \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{11-y}}{P(X < Y / Y=y)} \cdot P(Y=y)$$

$$= \frac{1}{2^{21}} \sum_{y=0}^{\infty} \sum_{x=0}^{y-1} 10C_x \cdot 11C_y$$

$$= \frac{1}{2^{21}} \sum_{y=0}^{\infty} \sum_{k=y}^{\infty} 10C_{y-k} \cdot 11C_y$$

$$= \frac{1}{2^{21}} \sum_{y=0}^{\infty} \sum_{k=1}^y 10C_{y-k} \cdot 11C_y = \frac{1}{2^{21}} \sum_{k=1}^{\infty} \sum_{y=k}^{\infty} 10C_{y-k} \cdot 11C_y$$

put $y-x=k$
 $x=y-k$
 @ $x=0$ $y=k$
 @ $x=y-1$ $k=1$

$X \sim \text{Bino}(10, 1/2)$
 $Y \sim \text{Bino}(11, 1/2)$

★ Given $X \sim N(\mu_1, \sigma_1^2)$ find $P(X > Y+k)$
 $Y \sim N(\mu_2, \sigma_2^2)$

Solution define $T = X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

$$P(T > k) = P\left(\frac{T - \mu_T}{\sigma_T} > \frac{k - \mu_T}{\sigma_T}\right)$$

$$= P\left(Z > \frac{k - \mu_T}{\sigma_T}\right)$$

$$= 1 - \Phi\left(\frac{k - \mu_T}{\sigma_T}\right)$$

$$\mu_T = \mu_1 - \mu_2$$

$$\sigma_T = \sqrt{\sigma_1^2 + \sigma_2^2}$$

Theorem

if $X \sim N(\mu_x, \sigma_x^2)$
 $Y \sim N(\mu_y, \sigma_y^2)$

- 1) $X+Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$
- 2) $-Y \sim N(-\mu_y, \sigma_y^2)$
- 3) $X-Y \sim N(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2)$

Random variables X and Y have a joint PMF given by $P_{X,Y}(x,y) = \begin{cases} \frac{1}{21} & x=0,1,2,3,4,5 \\ 0 & y=0,1,\dots,x \\ \text{otherwise} \end{cases}$

What is the expected value of X (i.e., $E[X]$)?

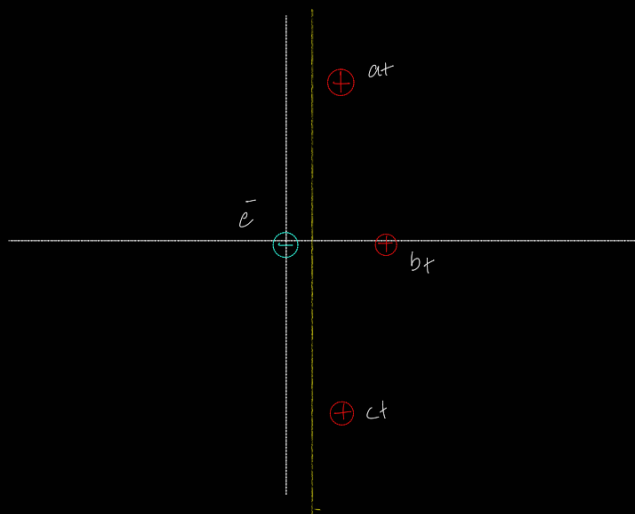
- A. $\frac{10}{3}$ ✓
- B. $\frac{5}{3}$
- C. $\frac{20}{3}$
- D. $\frac{10}{9}$

$$E(X) = \sum_{x=0}^5 \sum_{y=0}^x x \cdot \frac{1}{21}$$

$$E(X) = \frac{1}{21} \sum_{x=0}^5 (x+1) \cdot x$$

$$E(X) = \frac{1}{21} [1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + 5 \cdot 6]$$

$$= \frac{2+6+12+20+30}{21} = \frac{70}{21} = \frac{10}{3}$$



Variance & Expectation of Multivariate distribution

$$\begin{aligned}
 E(\vec{x}) &= \mu_x \\
 \text{Cov}(\vec{x}) &= E(\vec{x}\vec{x}^T) - E(\vec{x})E(\vec{x})^T \\
 \text{Cov}(\vec{x}) &= E(\vec{x}\vec{x}^T) - \mu_x \mu_x^T
 \end{aligned}
 \left. \begin{array}{l}
 \cdot \text{diagonal element Variance of } x_i \\
 \cdot \text{off diagonal element Cov}(x_i, x_j) \\
 \cdot \text{Symmetric Matrix}
 \end{array} \right\}$$

Transformation $\vec{s} = W\vec{x}$ for any matrix W

$$\begin{aligned}
 E(\vec{s}) &= WE(\vec{x}) \\
 \text{Cov}(\vec{s}) &= E[\vec{s}\vec{s}^T] - E[\vec{s}]E[\vec{s}]^T = E[W\vec{x}\vec{x}^TW^T] - WE[\vec{x}]E[\vec{x}]^TW^T \\
 &= W[E[\vec{x}\vec{x}^T] - E[\vec{x}]E[\vec{x}]^T]W^T \\
 \text{Cov}(\vec{s}) &= W\text{Cov}(\vec{x})W^T
 \end{aligned}$$

$$\begin{aligned}
 E[X|X>a] \\
 X \sim \text{Expo}(\lambda)
 \end{aligned}$$

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$P(X > a) = \int_a^\infty \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x} \right]_a^\infty = \underline{\underline{e^{-\lambda a}}}$$

$$f_X(x|X>a) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda a}}, \quad x > a$$

$$E(X|X>a) = \int_a^\infty x \cdot \frac{\lambda e^{-\lambda x}}{e^{-\lambda a}} dx = \frac{\lambda}{e^{-\lambda a}} \int_a^\infty x e^{-\lambda x} dx = \frac{\lambda}{e^{-\lambda a}} \left[x \frac{e^{-\lambda x}}{-\lambda} - \frac{1}{\lambda^2} e^{-\lambda x} \right]$$

$$= \frac{\lambda}{e^{-\lambda a}} \left[\frac{a e^{-\lambda a}}{\lambda} + \frac{e^{-\lambda a}}{\lambda^2} \right] = \frac{e^{-\lambda a}}{e^{-\lambda a}} \left[a + \frac{1}{\lambda} \right] = \underline{\underline{E[X] + a}}$$

Memorylessness
property of exponential

Sumner's law Good question

$$X \text{ is RV } \text{Var}(X)=2; f_{Y/X}(y/x) = \begin{cases} 4(y-x)^3 & x \leq y \leq x+1 \\ 0 & \text{elsewhere} \end{cases}$$

Find $\text{Var}(Y)$

$$\text{Var}(Y) = E[\text{Var}(Y/X)] + \text{Var}[E(Y/X)]$$

$$E[Y/X] = \int_x^{x+1} 4y(y-x)^3 dy \quad \left| \begin{array}{l} \text{put } y-x=z \\ dy=dz \\ y=x+z \end{array} \right.$$

$$= 4 \int_0^1 (x+z)^3 z^3 dz$$

$$= \left[\frac{4xz^4}{4} + \frac{4z^5}{5} \right]_0^1 = \boxed{x + \frac{4}{5}}$$

$$E[Y^2/X] = \int_x^{x+1} 4y^2(y-x)^3 dy \quad \begin{array}{l} y-x=z \\ dy=dz \end{array}$$

$$= \int_0^1 4(x+z)^2 z^3 dz$$

$$= \left[\frac{4x^2 z^4}{4} + \frac{4xz^5}{6} + \frac{8z^6}{5} \right]_0^1$$

$$= \boxed{x^2 + \frac{4}{6}x + \frac{8}{5}}$$

$$\text{Var}(Y/X) = E(Y^2/X) - E^2(Y/X)$$

$$= x^2 + \frac{4}{6}x + \frac{8}{5} - \left[x^2 + \frac{8}{5}x + \frac{16}{25} \right]$$

$$= \frac{4}{6} - \frac{16}{25} = \frac{4}{150} = \boxed{\frac{2}{75}}$$

$$\text{Var}(Y) = E\left[\frac{2}{75}\right] + \text{Var}\left[x + \frac{4}{5}\right] = \frac{2}{75} + 2 = \frac{152}{75} = \boxed{2.02}$$

Method 2
elegant

$$f_{Y/X} = \begin{cases} 4(y-x)^3 & x \leq y \leq x+1 \\ 0 & \text{otherwise} \end{cases}$$

let $Z = Y - X$ change of variable

$$f(z/x)(z/x) = 4z^3 \quad 0 \leq z \leq 1$$

→ Limit & function doesn't depend on x

$$\Rightarrow f_{Z/Y} = f_Z = 4z^3 \quad 0 \leq z \leq 1 \quad \forall x$$

→ Unconditional

$$x = g(y)$$

$$f_Y(y) = f_X(g(y)) \cdot \left| \frac{dx}{dy} \right|$$

→ Conditional

$$f_{Z/X}(z/x) = f_{Y/X}(g(z/x)) \cdot \left| \frac{dy}{dz} \right|$$

$$y = g(z) = z + x \quad (\text{here})$$

$$\frac{dy}{dz} = 1 \quad (\text{here})$$

$$Y = Z + X \quad Z \perp X$$

$$\text{Var}(Y) = \text{Var}(Z) + \text{Var}(X)$$

$$\text{Var}(Y) = \frac{2}{75} + 2 = \frac{152}{75} \approx \underline{\underline{2.02}}$$

$$\text{Var}(Z) = \int_0^1 4z^5 dz - \left[\int_0^1 4z^4 dz \right]^2 = \frac{4}{6} - \frac{16}{25} = \frac{4}{150} = \frac{2}{75}$$

Q) A chip has probability 0.05 of being defective independent of others. One Box has 6 chips. A box is labelled rejected if it contains atleast 2 defective chips. What is the probability of a container with 5 such Boxes having exactly 2 Box labelled rejected? (in percentage)

$$P(\text{acceptable Box}) = \frac{{}^6C_0 (0.05)^0 (0.95)^6 + {}^6C_1 0.05 \cdot (0.95)^5}{1} = 0.96722 = p$$

$$P(\text{exactly 2 Box defective}) = {}^5C_2 p^2 (1-p)^3 = 0.0097 = \underline{\underline{0.97\%}}$$

Q) $f_{X,Y}(x,y) = \begin{cases} 6(1-x-y) & 0 < x < 1, 0 < y < 1-x \\ 0 & \text{otherwise} \end{cases}$

Find Variance of $(X+Y)$?

let $z = x+y$
 $w = x-y$
 $\frac{z+w}{2} = x$
 $\frac{z-w}{2} = y$

$$f_{z,w}(z,w) = f_{x,y}(x,y) \cdot \left| J_{x,y \rightarrow z,w} \right|$$

$$\Rightarrow f_{z,w}(z,w) = 6(1-z) \cdot \frac{1}{2} = 3(1-z) \quad \begin{matrix} -z < w < z \\ 0 < z < 1 \end{matrix}$$

$$f_z(z) = \int_{-z}^z 3(1-z) dw = 6z(1-z) \quad 0 < z < 1$$

$$\text{Var}(X+Y) = \text{Var}(Z) = \int_0^1 6z^3 - 6z^4 dz - \left(\int_0^1 6z^2 - 6z^3 dz \right)^2$$

$$= \left(\frac{6}{4} - \frac{6}{5} \right) - \left(\frac{6}{3} - \frac{6}{4} \right)^2 = \frac{6}{20} - \left(\frac{6}{12} \right)^2 = \frac{6-5}{20} = \underline{\underline{\frac{1}{20}}}$$

$x > 0$	$y > 0$
$\frac{z+w}{2} > 0$	$\frac{z-w}{2} > 0$
$z+w > 0$	$z > w$
$w > -z$	

$\Rightarrow \boxed{-z < w < z}$

$y < 1-x$
 $z-w < 2-zw$
 $\boxed{z < 1}$

$x+y > 0 \Rightarrow \boxed{z > 0}$

$$J_{x,y \rightarrow z,w} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \underline{\underline{\frac{1}{2}}}$$

Easier Method

find $E(x) \ E(y) \ E(xy) \ E(x^2) \ E(y^2)$

then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$

$$\text{Var}(X+Y) = E(X^2) - E(X)^2 + E(Y^2) - E(Y)^2 + 2[E(XY) - E(X)E(Y)]$$

$$g) f_{X,Y}(x,y) = \begin{cases} 1, & -y \leq x \leq y; 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y=1) = \int_{-1}^1 1 \, dx = 2$$

$$\text{Find } E(X^2 | Y=1)$$

$$E(X^2 | Y=1) = \int_{-1}^1 x^2 \cdot \frac{f_{X,Y}(x,y)}{f_Y(y=1)} \, dx = \int_{-1}^1 x^2 \cdot \frac{1}{2} \, dx = \frac{x^3}{6} \Big|_{-1}^1 = \frac{2}{6} = \frac{1}{3} = \boxed{0.33}$$

Consider a sequence of independent Bernoulli trials with probability of success in each trial as $\frac{1}{3}$.
The probability that two successes occur before three failures is equal to

$I = \text{event } 2 \text{ S before } 3 \text{ F}$

$$P(S) = 1/3$$

$$S = \text{Head} = 1/3$$

Apply negative binomial directly

Here:

$$p = \frac{1}{3}, \quad q = \frac{2}{3} \quad \text{negn}(r, p)$$

For $r = 2$:

$$P(K = k) = \binom{1+k}{k} p^2 q^k$$

Now sum only the allowed values.

$$P(X=r) = \binom{k+r-1}{r} p^r q^k$$

$k = 0$	$P = \binom{1}{0} \left(\frac{1}{3}\right)^2 = \frac{1}{9}$
$k = 1$	$P = \binom{2}{1} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right) = \frac{4}{27}$
$k = 2$	$P = \binom{3}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = \frac{4}{27}$

$$P(I) = P(I|HH) \cdot P(HH) + P(I|THH) \cdot P(THH) + P(I|TTH) \cdot P(TTH) + P(I|HTH) \cdot P(HTH) + P(I|HTT) \cdot P(HTT) + P(I|THT) \cdot P(THT) + P(I|TTH) \cdot P(TTH) + P(I|THT) \cdot P(THT)$$

$$= 1 \cdot \frac{1}{9} + \frac{1}{3} \cdot \frac{4}{9} \cdot \frac{1}{3} + 0 + 1 \cdot \frac{1}{9} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{4}{9} + 1 \cdot \frac{2}{3} \cdot \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{4}{9} = \frac{3}{27} + \frac{4}{27} + \frac{4}{27} = \frac{11}{27}$$

$r = \# \text{Success}$
 $k = \# \text{failures}$

$P(k) = \text{Probability of } k \text{ failures before } r \text{ Success}$

$P(k) \sim \text{Negative Binomial}(2, 1/3)$

$$P(k) = \binom{k+r-1}{k} p^r (1-p)^k$$

$$P(2 \text{ S before } 3 \text{ failure}) = P(0) + P(1) + P(2)$$

$$= \binom{1}{0} \left(\frac{1}{3}\right)^2 + \binom{2}{1} \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} + \binom{3}{2} \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{9} + 2 \cdot \frac{2}{27} + 3 \cdot \frac{1}{9} \cdot \frac{4}{9}$$

$$= \frac{3+4+4}{27} = \boxed{\frac{11}{27}}$$